Ring Theory 13

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Q> R, δ is a Euclidean pair. There exist a subring $S \subseteq R$
Let $a \in S$ and $a \notin R^* \cup \{0\}$. Given $b \in S - \{0\}$, $\exists$ a unique $n \in \mathbb{N}$ and unique $r_0, r_1, \ldots, r_n \in \mathbb{N}$ and $r_i \neq 0$ $\forall i \in \{0, \ldots, n\}$ and $b = r_0 + r_1 a + r_2 a^2 + \cdots + r_n a^n$

$\longrightarrow$ to be done latter

Modules:-

Def:- Module is A triplet $(M, +, \cdot)$ where $(M, +)$ is an abelian group and $\cdot$ is a map from $R \times M$ to M, satisfying:

(i) $a \cdot (m + m') = a \cdot m + a \cdot m' \quad \forall\, m, m' \in M, a \in R$

(ii) $(a + b) \cdot m = a \cdot m + b \cdot m \quad \forall\, a, b \in R, m \in M$

(iii) $a \cdot (b \cdot m) = (a \cdot b) \cdot m \quad \forall\, m \in M, a, b \in R$

(iv) $1 \cdot m = m \quad \forall\, m \in M$

•> A subgroup $N \leq M$ is called a sub module of M, if it is closed under multiplication (as in M), i.e., $\forall\, a \in R, m \in N$, $am \in N$

•> M/N is the quotient group where, $a \cdot \bar{m} = \overline{a m}$ $\forall\, \bar{m} \in M/N, a \in R$

Def:- (Homomorphism of modules)

Let M, M' be modules over R. Then the function,

$$f : M \to M' \text{ is a homomorphism if:-}$$

(i) f is a group homomorphism

(ii) f preserves scalar multiplication, $f(am) = a f(m)$ $\forall\, a \in R, m \in M$

- A bijective homomorphism is a isomorphism

- A vector space is a module over a field
- Ideal of a ring is a module. (R is a module over itself as well)
- Every abelian group is a module over the ring of integers ($\mathbb{Z}$).

$\mathbb{Z} \times G \longrightarrow G$

$(n, g) \longrightarrow ng = \underbrace{g + g + g + \cdots + g}_{n \text{ times}}$ $\qquad g = \mathbb{Z}/n\mathbb{Z}$ is 3 $\rightarrow$ no basis

Q> If $f: M \rightarrow M'$ is an isomorphism of R-modules, prove that $f^{-1}: M' \rightarrow M$ is also a isomorphism of R-modules.

Ans:- $b_1, b_2 \in M'$ $\quad \exists a_1, a_2 \in M$, $f(a_1) = b_1$ and $f(a_2) = b_2$

$f^{-1}(b_1 + b_2) = f^{-1}(f(a_1) + f(a_2)) = f^{-1}(f(a_1 + a_2))$

$= a_1 + a_2 = f^{-1}(b_1) + f^{-1}(b_2)$

$f^{-1}(a b_1) = f^{-1}(a f(a_1)) = f^{-1}(f(a a_1)) = a a_1 = a f^{-1}(b_1)$

Q> Prove that a natural map, $f: M \longrightarrow M/N$ is an R-module homomorphism.

Ans:- $f(a_1 + a_2) = (a_1 + a_2) + N$

$= (a_1 + N) + (a_2 + N) = f(a_1) + f(a_2)$

Q> Let R be a module over itself, then, prove that a singleton set $\{x\}$ is linearly independent iff x is not a zero divisor in R singleton set {x} is linearly independent iff x is not a zero divisor in R

Ans:- Let $\{x\}$ to be linearly independent. If $rx = 0 \Rightarrow r = 0$

$\Rightarrow$ x is not a zero divisor

If x is not zero divisor $\Rightarrow \forall r \in R,\ rx \neq 0$

Let $a \neq 0, a \in R$ such that $ax \neq 0$ and if is for all $a \in R, a \neq 0$

$\Rightarrow$ $\{x\}$ is linearly independent

Modules over Commutative Rings:-

- Basis for every modules may not exist over comm. ring
- Linearly independent subset of a module cannot be completed to a basis for modules in comm. ring.
- S is a maximal linearly independent set $\Longleftrightarrow$ S is minimal system of generators $\Longleftrightarrow$ S is a basis

$\Downarrow$

This is not true for modules but it is for vector space

Def:- (Free modules)

An R-module M is said to be free when $\exists$ a basis for M

- Free modules are like vector spaces.
- Any vector space over a field is free.
- A submodule of a free module may not be free

$\mathbb{Z}/6\mathbb{Z}$, $\quad 2(\mathbb{Z}/6\mathbb{Z}) \subseteq \mathbb{Z}/6\mathbb{Z}$ $\quad$ 0, 2, 4, 6, 8, 10

$\longrightarrow$ is not free $\quad$ is 3. these $\equiv 0$

• If an ideal $I \subseteq R$ is free as an R-module, then I is a principal ideal. A principal ideal I is free if it is generated by a non zero divisor. In general if R is an integral domain, then an ideal is free iff it is principal. (R is comm.)